



MARKSCHEME

November 2011

MATHEMATICS

Higher Level

Paper 1

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Instructions to Examiners

Abbreviations

- M** Marks awarded for attempting to use a correct **Method**; working must be seen.
- (M)** Marks awarded for **Method**; may be implied by **correct** subsequent working.
- A** Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding **M** marks.
- (A)** Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.
- R** Marks awarded for clear **Reasoning**.
- N** Marks awarded for **correct** answers if **no** working shown.
- AG** Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Write the marks in red on candidates' scripts, in the right hand margin.

- Show the **breakdown** of individual marks awarded using the abbreviations **MI**, **AI**, etc.
- Write down the total for each **question** (at the end of the question) and **circle** it.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award **M0** followed by **AI**, as **A** mark(s) depend on the preceding **M** mark(s), if any.
- Where **M** and **A** marks are noted on the same line, e.g. **MIAI**, this usually means **MI** for an **attempt** to use an appropriate method (e.g. substitution into a formula) and **AI** for using the **correct** values.
- Where the markscheme specifies (**M2**), **N3**, etc., do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 N marks

*Award N marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of **N** and other marks.
- There may be fewer **N** marks available than the total of **M**, **A** and **R** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets e.g. (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer **FT** marks.
- If the error leads to an inappropriate value (e.g. $\sin \theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent A** marks can be awarded, but **M** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). Apply a **MR** penalty of 1 mark to that question. Award the marks as usual and then write $-1(\mathbf{MR})$ next to the total. Subtract 1 mark from the total for the question. A candidate should be penalized only once for a particular mis-read.*

- If the question becomes much simpler because of the **MR**, then use discretion to award fewer marks.
- If the **MR** leads to an inappropriate value (e.g. $\sin \theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks (*d*)

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. The mark should be labelled (**d**) and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, etc.
- Alternative solutions for part-questions are indicated by **EITHER . . . OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x - 3)$, the markscheme gives:

$$f'(x) = (2\cos(5x - 3))5 \quad (= 10\cos(5x - 3)) \quad \text{AI}$$

Award **AI** for $(2\cos(5x - 3))5$, even if $10\cos(5x - 3)$ is not seen.

10 Accuracy of Answers

The method of dealing with accuracy errors on a whole paper basis by means of the Accuracy Penalty (**AP**) no longer applies.

Instructions to examiners about such numerical issues will be provided on a question by question basis within the framework of mathematical correctness, numerical understanding and contextual appropriateness.

The rubric on the front page of each question paper is given for the guidance of candidates. The markscheme (**MS**) may contain instructions to examiners in the form of “Accept answers which round to n significant figures (**sf**)”. Where candidates state answers, required by the question, to fewer than n **sf**, award **A0**. Some intermediate numerical answers may be required by the **MS** but not by the question. In these cases only award the mark(s) if the candidate states the answer exactly or to at least 2**sf**.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

No calculator is allowed. The use of any calculator on paper 1 is malpractice, and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice. Examples: finding an angle, given a trig ratio of 0.4235.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

SECTION A

1. area of triangle $= \frac{1}{2}(2x)^2 \sin \frac{\pi}{3}$ *(MI)*
 $= x^2 \sqrt{3}$ *AI*

Note: A $0.5 \times \text{base} \times \text{height}$ calculation is acceptable.

area of sector $= \frac{\theta}{2} r^2 = \frac{\pi}{6} r^2$ *(MI)AI*

area of triangle is twice the area of the sector

$\Rightarrow 2\left(\frac{\pi}{6} r^2\right) = x^2 \sqrt{3}$ *MI*

$\Rightarrow r = x \sqrt{\frac{3\sqrt{3}}{\pi}}$ or equivalent *AI*

[6 marks]

2. $i = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$ *(AI)*

$z_1 = i^{\frac{1}{3}} = \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)^{\frac{1}{3}} = \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \left(= \frac{\sqrt{3}}{2} + \frac{1}{2}i \right)$ *MIAI*

$z_2 = \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \left(= -\frac{\sqrt{3}}{2} + \frac{1}{2}i \right)$ *(MI)AI*

$z_3 = \cos \left(-\frac{\pi}{2}\right) + i \sin \left(-\frac{\pi}{2}\right) = -i$ *AI*

Note: Accept exponential and cis forms for intermediate results, but not the final roots.

Note: Accept the method based on expanding $(a + b)^3$. *MI* for attempt, *MI* for equating real and imaginary parts, *AI* for finding $a = 0$ and $b = \frac{1}{2}$, then *AI**AI**AI* for the roots.

[6 marks]

3. tree diagram

(MI)

$$P(I|D) = \frac{P(D|I) \times P(I)}{P(D)}$$

(MI)

$$= \frac{0.1 \times 0.2}{0.1 \times 0.2 + 0.8 \times 0.75}$$

AIAIAI

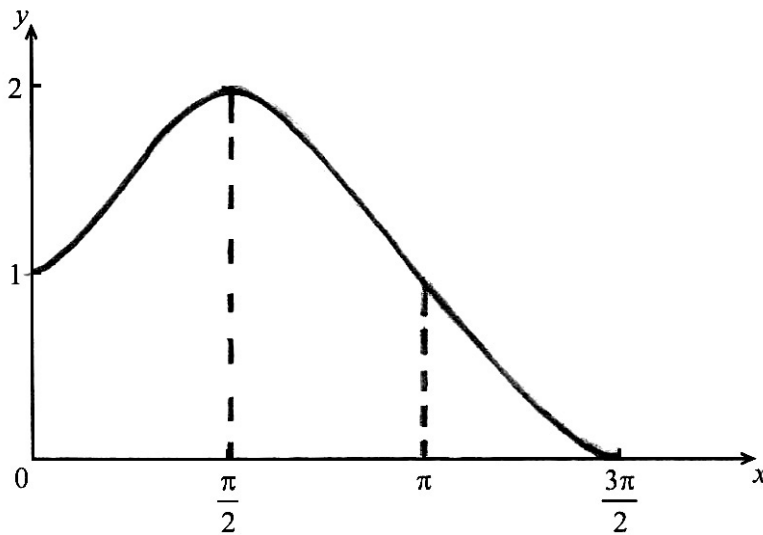
$$\left(= \frac{0.02}{0.62} \right) = \frac{1}{31}$$

AI

Note: Alternative presentation of results: *MI* for labelled tree; *AI* for initial branching probabilities, 0.2 and 0.8; *AI* for at least the relevant second branching probabilities, 0.1 and 0.75; *AI* for the ‘infected’ end-point probabilities, 0.02 and 0.6; *MIAI* for the final conditional probability calculation.

[6 marks]

4. (a)



AI

(b) $(1 + \sin x)^2 = 1 + 2\sin x + \sin^2 x$

$$= 1 + 2\sin x + \frac{1}{2}(1 - \cos 2x)$$

AI

$$= \frac{3}{2} + 2\sin x - \frac{1}{2}\cos 2x$$

AG

continued ...

Question 4 continued

$$(c) \quad V = \pi \int_0^{\frac{3\pi}{2}} (1 + \sin x)^2 dx \quad (M1)$$

$$= \pi \int_0^{\frac{3\pi}{2}} \left(\frac{3}{2} + 2 \sin x - \frac{1}{2} \cos 2x \right) dx$$

$$= \pi \left[\frac{3}{2}x - 2 \cos x - \frac{\sin 2x}{4} \right]_0^{\frac{3\pi}{2}} \quad A1$$

$$= \frac{9\pi^2}{4} + 2\pi \quad A1A1$$

[6 marks]

$$5. \quad P(A) = \frac{\pi}{25\pi} \times \frac{1}{2} = \frac{1}{50} \quad (M1)A1$$

$$P(B) = \frac{8\pi}{25\pi} \times \frac{1}{2} = \frac{4}{25} \quad A1$$

$$P(C) = \frac{16\pi}{25\pi} \times \frac{1}{2} = \frac{8}{25} \quad A1$$

Note: The *MI* is for the use of 3 areas

$$E(X) = (0.5 \times 0) + \frac{1}{50} \times 10 + \frac{4}{25} \times 6 + \frac{8}{25} \times 3 = \frac{106}{50} (= 2.12) \quad M1A1$$

Note: The final *MI* is available if the probabilities are incorrect but sum to 1 or

[6 marks]

6. proposition is true for $n = 1$ since $\frac{dy}{dx} = \frac{1}{(1-x)^2}$ *MI*
 $= \frac{1!}{(1-x)^2}$ *AI*

Note: Must see the 1! for the *AI*.

assume true for $n = k$, $k \in \mathbb{Z}^+$, i.e. $\frac{d^k y}{dx^k} = \frac{k!}{(1-x)^{k+1}}$ *MI*

consider $\frac{d^{k+1}y}{dx^{k+1}} = \frac{d\left(\frac{d^k y}{dx^k}\right)}{dx}$ *(MI)*
 $= (k+1)k!(1-x)^{-(k+1)-1}$ *AI*
 $= \frac{(k+1)!}{(1-x)^{k+2}}$ *AI*

hence, P_{k+1} is true whenever P_k is true, and P_1 is true, and therefore the proposition is true for all positive integers *RI*

Note: The final *RI* is only available if at least 4 of the previous marks have been awarded.

[7 marks]

7. to find the points of intersection of the two curves *MI*
 $-x^2 + 2 = x^3 - x^2 - bx + 2$
 $x^3 - bx = x(x^2 - b) = 0$
 $\Rightarrow x = 0$; $x = \pm\sqrt{b}$ *AIAI*
 $A_1 = \int_{-\sqrt{b}}^0 [(x^3 - x^2 - bx + 2) - (-x^2 + 2)] dx \left(= \int_{-\sqrt{b}}^0 (x^3 - bx) dx \right)$ *MI*
 $= \left[\frac{x^4}{4} - \frac{bx^2}{2} \right]_{-\sqrt{b}}^0$
 $= - \left(\frac{(-\sqrt{b})^4}{4} - \frac{b(-\sqrt{b})^2}{2} \right) = - \frac{b^2}{4} + \frac{b^2}{2} = \frac{b^2}{4}$ *AI*
 $A_2 = \int_0^{\sqrt{b}} [(-x^2 + 2) - (x^3 - x^2 - bx + 2)] dx$ *MI*

continued ...

Question 7 continued

$$= \int_0^{\sqrt{b}} (-x^3 + bx) dx$$

$$= \left[-\frac{x^4}{4} + \frac{bx^2}{2} \right]_0^{\sqrt{b}} = \frac{b^2}{4}$$

AI

therefore $A_1 = A_2 = \frac{b^2}{4}$

AG

[7 marks]

8. (a) angle APB is a right angle

$$\Rightarrow \cos \theta = \frac{AP}{4} \Rightarrow AP = 4 \cos \theta$$

AI

Note: Allow correct use of cosine rule.

$$\text{arc PB} = 2 \times 2\theta = 4\theta$$

AI

$$t = \frac{AP}{3} + \frac{PB}{6}$$

MI

Note: Allow use of their AP and their PB for the MI.

$$\Rightarrow t = \frac{4 \cos \theta}{3} + \frac{4\theta}{6} = \frac{4 \cos \theta}{3} + \frac{2\theta}{3} = \frac{2}{3}(2 \cos \theta + \theta)$$

AG

(b) $\frac{dt}{d\theta} = \frac{2}{3}(-2 \sin \theta + 1)$

AI

$$\frac{2}{3}(-2 \sin \theta + 1) = 0 \Rightarrow \sin \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6} \text{ (or 30 degrees)}$$

AI

(c) $\frac{d^2t}{d\theta^2} = -\frac{4}{3} \cos \theta < 0 \left(\text{at } \theta = \frac{\pi}{6} \right)$

MI

$$\Rightarrow t \text{ is maximized at } \theta = \frac{\pi}{6}$$

RI

time needed to walk along arc AB is $\frac{2\pi}{6}$ (≈ 1 hour)

time needed to row from A to B is $\frac{4}{3}$ (≈ 1.33 hour)

hence, time is minimized in walking from A to B

RI

[8 marks]

9. (a) for the equation to have real roots
 $(y-1)^2 - 4y(y-1) \geq 0$ *MI*
- $\Rightarrow 3y^2 - 2y - 1 \leq 0$
(by sign diagram, or algebraic method) *MI*
- $-\frac{1}{3} \leq y \leq 1$ *AIAI*

Note: Award first *AI* for $-\frac{1}{3}$ and 1, and second *AI* for inequalities.
These are independent marks.

- (b) $f : x \rightarrow \frac{x+1}{x^2+x+1} \Rightarrow x+1 = yx^2 + yx + y$ *(MI)*
- $\Rightarrow 0 = yx^2 + (y-1)x + (y-1)$ *AI*
- hence, from (a) range is $-\frac{1}{3} \leq y \leq 1$ *AI*
- (c) a value for y would lead to 2 values for x from (a) *RI*

Note: Do not award *RI* if (b) has not been tackled.

[8 marks]

SECTION B

10. (a) $k \int_0^{\frac{\pi}{2}} \sin x \, dx = 1$ *MI*
 $k [-\cos x]_0^{\frac{\pi}{2}} = 1$
 $k = 1$ *AI*

[2 marks]

(b) $E(X) = \int_0^{\frac{\pi}{2}} x \sin x \, dx$ *MI*
integration by parts *MI*
 $[-x \cos x]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x \, dx$ *AIAI*
 $= 1$ *AI*

[5 marks]

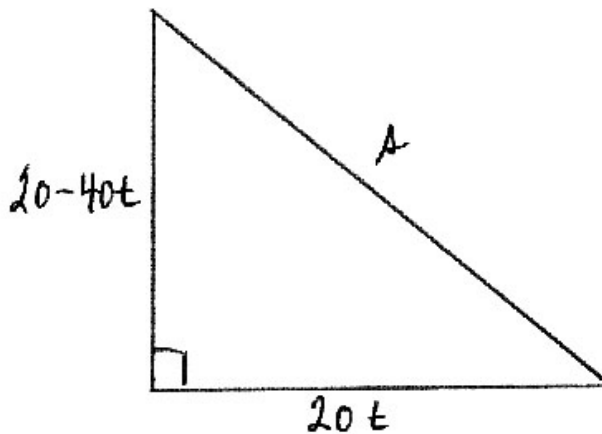
(c) $\int_0^M \sin x \, dx = \frac{1}{2}$ *MI*
 $[-\cos x]_0^M = \frac{1}{2}$ *AI*
 $\cos M = \frac{1}{2}$
 $M = \frac{\pi}{3}$ *AI*

Note: accept $\arccos \frac{1}{2}$

[3 marks]

Total [10 marks]

11. (a)



(M1)

$$s^2 = (20t)^2 + (20 - 40t)^2$$

M1

$$s^2 = 2000t^2 - 1600t + 400$$

A1

to minimize s it is enough to minimize s^2

$$f'(t) = 4000t - 1600$$

A1

setting $f'(t)$ equal to 0

M1

$$4000t - 1600 = 0 \Rightarrow t = \frac{2}{5} \text{ or } 24 \text{ minutes}$$

A1

$$f''(t) = 4000 > 0$$

M1

\Rightarrow at $t = \frac{2}{5}$, $f(t)$ is minimized

hence, the ships are closest at 12:24

A1

Note: accept solution based on s .

[8 marks]

(b) $f\left(\frac{2}{5}\right) = \sqrt{80}$

M1A1

since $\sqrt{80} < 9$, the captains can see one another

R1

[3 marks]

Total [11 marks]

12. (a) (i) $|a - b| = |a + b|$
 $\Rightarrow (a - b) \cdot (a - b) = (a + b) \cdot (a + b)$ (MI)
 $\Rightarrow |a|^2 - 2a \cdot b + |b|^2 = |a|^2 + 2a \cdot b + |b|^2$ AI
 $\Rightarrow 4a \cdot b = 0 \Rightarrow a \cdot b = 0$ AI
 therefore a and b are perpendicular RI

Note: Allow use of 2-d components.

Note: Do not condone sloppy vector notation, so we must see something to the effect that $|c|^2 = c \cdot c$ is clearly being used for the **MI**.

Note: Allow a correct geometric argument, for example that the diagonals of a parallelogram have the same length only if it is a rectangle.

- (ii) $|a \times b|^2 = (|a||b|\sin\theta)^2 = |a|^2|b|^2\sin^2\theta$ MIAI
 $|a|^2|b|^2 - (a \cdot b)^2 = |a|^2|b|^2 - |a|^2|b|^2\cos^2\theta$ MI
 $= |a|^2|b|^2(1 - \cos^2\theta)$ AI
 $= |a|^2|b|^2\sin^2\theta$
 $\Rightarrow |a \times b|^2 = |a|^2|b|^2 - (a \cdot b)^2$ AG

[8 marks]

- (b) (i) area of triangle $= \frac{1}{2} |\vec{AB} \times \vec{AC}|$ (MI)
 $= \frac{1}{2} |(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})|$ AI
 $= \frac{1}{2} |\mathbf{b} \times \mathbf{c} + \mathbf{b} \times -\mathbf{a} + -\mathbf{a} \times \mathbf{c} + -\mathbf{a} \times -\mathbf{a}|$ AI
 $\mathbf{b} \times -\mathbf{a} = \mathbf{a} \times \mathbf{b}; \mathbf{c} \times \mathbf{a} = -\mathbf{a} \times \mathbf{c}; -\mathbf{a} \times -\mathbf{a} = 0$ MI
 hence, area of triangle is $\frac{1}{2} |\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}|$ AG

- (ii) D is the foot of the perpendicular from B to AC
 area of triangle ABC $= \frac{1}{2} |\vec{AC}| |\vec{BD}|$ AI
 therefore
 $\frac{1}{2} |\vec{AC}| |\vec{BD}| = \frac{1}{2} |\vec{AB} \times \vec{AC}|$ MI
 hence, $|\vec{BD}| = \frac{|\vec{AB} \times \vec{AC}|}{|\vec{AC}|}$ AI
 $= \frac{|\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}|}{|\mathbf{c} - \mathbf{a}|}$ AG

[7 marks]

Total [15 marks]

13. (a) $\frac{dy}{dx} = \frac{e}{\ln e}(2+2) = 4e$ *AI*
 at (2, e) the tangent line is $y - e = 4e(x - 2)$ *MI*
 hence $y = 4ex - 7e$ *AI*
- [3 marks]**
- (b) $\frac{dy}{dx} = \frac{y}{\ln y}(x+2) \Rightarrow \frac{\ln y}{y} dy = (x+2) dx$ *MI*
 $\int \frac{\ln y}{y} dy = \int (x+2) dx$
 using substitution $u = \ln y$; $du = \frac{1}{y} dy$ *(MI)(AI)*
 $\Rightarrow \int \frac{\ln y}{y} dy = \int u du = \frac{1}{2} u^2$ *(AI)*
 $\Rightarrow \frac{(\ln y)^2}{2} = \frac{x^2}{2} + 2x + c$ *AIAI*
 at (2, e), $\frac{(\ln e)^2}{2} = 6 + c$ *MI*
 $\Rightarrow c = -\frac{11}{2}$ *AI*
 $\Rightarrow \frac{(\ln y)^2}{2} = \frac{x^2}{2} + 2x - \frac{11}{2} \Rightarrow (\ln y)^2 = x^2 + 4x - 11$
 $\ln y = \pm \sqrt{x^2 + 4x - 11} \Rightarrow y = e^{\pm \sqrt{x^2 + 4x - 11}}$ *MIAI*
 since $y > 1$, $f(x) = e^{\sqrt{x^2 + 4x - 11}}$ *RI*
- [11 marks]**

Note: *MI* for attempt to make y the subject.

- (c) **EITHER**
- $x^2 + 4x - 11 > 0$ *AI*
 using the quadratic formula *MI*
 critical values are $\frac{-4 \pm \sqrt{60}}{2} (= -2 \pm \sqrt{15})$ *AI*
 using a sign diagram or algebraic solution *MI*
 $x < -2 - \sqrt{15}$; $x > -2 + \sqrt{15}$ *AIAI*
- OR**
- $x^2 + 4x - 11 > 0$ *AI*
 by methods of completing the square *MI*
 $(x+2)^2 > 15$ *AI*
 $\Rightarrow x+2 < -\sqrt{15}$ or $x+2 > \sqrt{15}$ *(MI)*
 $x < -2 - \sqrt{15}$; $x > -2 + \sqrt{15}$ *AIAI*

[6 marks]
 continued ...

Question 13 continued

$$(d) \quad f(x) = f'(x) \Rightarrow f(x) = \frac{f(x)}{\ln f(x)}(x+2)$$

MI

$$\Rightarrow \ln(f(x)) = x+2 \quad \left(\Rightarrow x+2 = \sqrt{x^2 + 4x - 11} \right)$$

AI

$$\Rightarrow (x+2)^2 = x^2 + 4x - 11 \Rightarrow x^2 + 4x + 4 = x^2 + 4x - 11$$

AI

$$\Rightarrow 4 = -11, \text{ hence } f(x) \neq f'(x)$$

RIAG

[4 marks]

Total [24 marks]
